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ABSTRACT

This document contains three modules. The first of these examines applications of algebra to geometry. It is designed to teach students how to algebraically characterize points which may be constructed with a compass and straight edge, and how to use this characterization to obtain classical geometric nonconstructibility results. The second unit features applications of statistics. It is designed to help the student: 1) perform the t-test on appropriate experimental data and interpret the calculated value of "t"; 2) understand the role of a statistical test of significance in the research process; and 3) recognize the relationship between the statistical arithmetic and the design and conduct of the experimental investigation. The final module views applications of probability. The student is taught to: 1) use the Monte Carlo technique to simulate simple experiments; 2) better appreciate the role of approximate solutions to complex problems. Each of the three modules includes exercises, and answers are provided. The second and third units contain model exams, with answer keys included. (JN)

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UMAP

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

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MODULE 267

The Impossibility of Trisecting Angles

by Mark D. Meyerson

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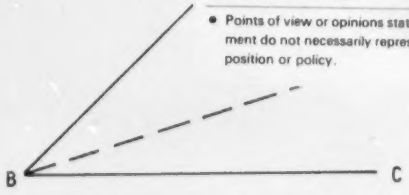

Applications of Algebra to Geometry

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THE IMPOSSIBILITY OF TRISECTING ANGLES

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Intermodular Description Sheet: UMAP Unit 267

Title: THE IMPOSSIBILITY OF TRISECTING ANGLES

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Review Stage/Date: IV 4/30/80

Classification: APPL GOEMETRY

References:

Moise, E.E., Elementary Geometry from an Advanced Standpoint, 2nd ed., Addison-Wesley, Reading, MA, 1974.

Peressini, A.L. and D.R. Sherbert, Topics in Modern Mathematics for Teachers, Holt, Rinehart, and Winston, NY, 1971.

Prerequisite Skills:

1. Elementary trigonometry.
2. General equation of a circle.
3. Manipulation of polynomials.
4. Fundamental theorem of algebra.
5. Elementary Euclidean geometry.

Output Skills:

1. To algebraically characterize those points which may be constructed with compass and straightedge.
2. To use this characterization to obtain classical geometric non-constructibility results.

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1. INTRODUCTION

One of the most intriguing geometrical problems of antiquity is to trisect an angle using a compass and straightedge. Although E. Galois proved (around 1830) that it is impossible, in general, to trisect an angle, much effort has since been wasted in futile constructions. Our goal is to give a brief and elementary proof of this nonconstructability. A few related theorems, such as the impossibility of duplicating the cube, are also included.

You might be surprised by all the algebra used in proving these geometric facts. The necessity of approaching these problems algebraically is the reason they were unsolved for so long. In fact, the most striking discoveries in mathematics often result from interplay between apparantly unrelated fields, that is, the application of one branch of mathematics to another branch.

The only background needed for the following material is some elementary high school mathematics, such as factoring polynomials, knowing the equation of a circle, and using some trigonometry.

Here is a precise statement of the problem. Given an arbitrary angle, $\angle ABC$, one would like to construct a point D with the measure of $\angle DBC$ one-third the measure of $\angle ABC$. All construction must be done only with compass and straightedge. Given two points E and F, a compass may only be used to draw the circle through E with center F and straightedge may only be used to draw the line through E and F. Points are constructed by intersecting a line or circle with another line or circle. Although certain angles, such as a 90° angle, can be trisected in this manner, we shall see that other angles, such as 60°, cannot be so trisected.

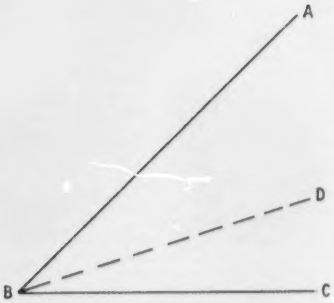

Figure 1. Angle DBC has one third the measure of angle ABC.

The sources for most of this module are the two books, Elementary Geometry from an Advanced Standpoint, by Edwin E. Moise, and Topics in Modern Mathematics for Teachers, by Anthony L. Peressini and Donald R. Sherbert. These books are recommended if you desire to continue with the subject.

2. SUBFIELDS

All our calculations will be done with real numbers. The set of real numbers is denoted by $\mathbb{R}$.

Definition 1. A subset, F, of $\mathbb{R}$, is called a *subfield* (of $\mathbb{R}$) if it contains 0 and 1, and if it is closed under division by non-zero elements of F and subtraction. For example, closed under subtraction means that if a and b are elements of F, so is a - b. Note that a subfield is closed under multiplication and addition, since $ab = a/(1/b)$ and $a + b = a - (0 - b)$. There is a technical definition of *field* which we shall not need.

Examples: 1. $\mathbb{R}$ is a subfield.

2. A number is called *rational* if it can be written as p/q for p and q ($\neq 0$) integers. The set of rational numbers is denoted $\mathbb{Q}$. We show in the aside below that $\mathbb{Q} \neq \mathbb{R}$. But $\mathbb{Q}$ is a subfield, since $0 = 0/1$, $1 = 1/1$, $(p/q)/(r/s) = (ps)/(rq)$ for $r/s \neq 0$ (hence $r \neq 0$), and $p/q - r/s = (ps - qr)/(qs)$.

3. The set of integers is not a subfield, since 1/2 is not an integer.

Aside. $\sqrt{2}$ is not rational.

Proof. Suppose $\sqrt{2}$ is rational. Then we could write it as p/q in reduced form. So $\sqrt{2}q = p$, and squaring, $2q^2 = p^2$.

Since p^2 is even, p must be even. So $p = 2m$ for some integer m. Substituting, we get $2q^2 = (2m)^2 = 4m^2$, or $q^2 = 2m^2$.

Since q^2 is even, q must be even. So p/q is *not* in reduced form, because p and q each have a factor of 2.

Hence $\sqrt{2}$ cannot be a rational number. □

We close this section with a theorem about the roots of an equation.

Theorem 1 (The Rational Root Test). If $a_n x^n + \ldots + a_1 x + a_0 = 0$ is a polynomial equation with integer coefficients and p/q is a rational root, in reduced form, then p divides a_0 and q divides a_n.

Proof: We have $a_n(p/q)^n + a_{n-1}(p/q)^{n-1} + \ldots + a_1(p/q) + a_0 = 0$. Multiplying by q^n we get $a_n p^n + a_{n-1}p^{n-1}q + \ldots + a_1 pq^{n-1} + a_0 q^n = 0$. Since p and q each divide the right hand side of this equation, they each divide the left hand side. And since p divides each term on the left, except perhaps $a_0 q^n$, p must divide $a_0 q^n$ also. But p and q have no factors in common, so p divides a_0. Similarly, q divides $a_n p^n$, and so divides a_n. □

3. SURDS

Definition 2. A number is called a *surd* if it can be calculated from 0 and 1 by a finite number of additions, subtractions, multiplications, divisions, and *extractions of square roots.*

Any rational number is a surd. And $\sqrt{\sqrt{2} + 1} - 3/2$ is a surd. There are many numbers which are not surds. We will see later that $\sqrt[3]{2}$ and $\cos(20^\circ)$ are not surds; also, π is not a surd.

The set of all surds forms a subfield. For 0 and 1 are surds, and if a,b and $c \neq 0$ are surds, so are $a - b$ and a/c.

We now consider the Euclidean plane with a coordinate system.

Definition 3. A *surd-curve* is a circle or line with equation $A(x^2 + y^2) + Dx + Ey + F = 0$, such that all the coefficients are surds. We may assume that $A = 1$ for a circle (why?) and $A = 0$ for a line. A *surd-point* is a point (x,y) such that x and y are surds.

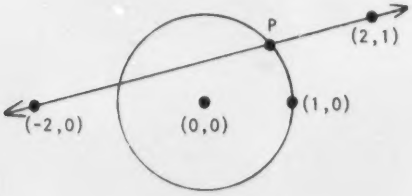

Figure 2. An example of two surd-curves.
$P = ((-2 + 4\sqrt{13})/17, (8 + \sqrt{13})/17)$ is a surd-point.

Theorem 2. If P = (x,y) lies on two distinct surd-curves, then P is a surd-point.

Proof: This can be proven by solving for P, and showing that x and y are surds. We prove only the hardest case, in which both surd-curves are circles.

The two surd-curves have equations $x^2 + y^2 + Dx + Ey + F = 0$ and $x^2 + y^2 + Gx + Hy + I = 0$, with surd coefficients. Subtracting, we get $Jx + Ky + L = 0$, where

J, K, and L are surds. J and K are not both zero, since if they were we would have distinct concentric circles meeting at P.

We now suppose $K \neq 0$. The proof is entirely analogous if $J \neq 0$. So we can solve for y, $y = Mx + N$, where M and N are surds. Subsituting into the very first equation, we get $ax^2 + bx + c = 0$, where a, b and c are surds. Since $a = 1 + M^2$, $a \neq 0$. So $x = (-b \pm \sqrt{b^2 - 4ac})/(2a)$ and $y = Mx +$, both surds. □

Theorem 3. Given a collection of only surd-points, any point we can construct using compass and straightedge must be a surd-point.

Proof: Let $P = (a,b)$ and $Q = (c,d)$ be surd-points.

It's easy to check directly that the line through P and Q has equation: $(d - b)x + (a - c)y + (bc - ad) = 0$, and that the circle with center P through Q has equation $x^2 + y^2 - 2ax - 2by + (2ac + 2bd - c^2 - d^2) = 0$. All coefficients are surds!

So only surd-curves can be constructed from surd-points. The only way to construct a new point is to consider the intersection of two of these surd-curves, which must be a surd-point by Theorem 2. We can continue constructing curves and points, but only surd-curves and surd-points. □

4. CUBIC EQUATIONS

Definition 4. Let F be a subfield (of $\mathbb{R}$) and let k be a positive number in F such that $\sqrt{k}$ is not in F. Then F(k) denotes the set of all numbers of the form $x + y\sqrt{k}$, where x and y are in F.

For example, if $F = \mathbb{Q}$, $k = 2$, we get $\mathbb{Q}(2)$, which includes $3 + 2\sqrt{2}$, $(1/2) - \sqrt{2}$, and $3 = 3 + 0\cdot\sqrt{2}$. Or if $F = \mathbb{Q}(2)$, $k = 3$, we get $F(3) = (\mathbb{Q}(2))(3)$ (we shall see in an exercise that $\sqrt{3} \notin \mathbb{Q}(2)$).

Each element of F(k) can be written as $x + y\sqrt{k}$ in only one way. For if $a + b\sqrt{k} = c + d\sqrt{k}$, then $(a - c) = (d - b)\sqrt{k}$. If $b \neq d$, then $\sqrt{k} = (a - c)/(d - b)$ an element of F, contradicting the choice of k. So $b = d$, and hence $a = c$.

Also F(k) is a subfield; let's check the definition of subfield. Now $0 = 0 + 0\cdot\sqrt{k}$ and $1 = 1 + 0\cdot\sqrt{k}$ are in F(k), and $(a + b\sqrt{k}) - (c + d\sqrt{k}) = (a - c) + (b - d)\sqrt{k}$ an element of F(k). So we only need check closure under division by non-zero elements. But

$$\frac{a+b\sqrt{k}}{c+d\sqrt{k}} = \frac{(a+b\sqrt{k})(c-d\sqrt{k})}{(c+d\sqrt{k})(c-d\sqrt{k})} = \left(\frac{ac-bdk}{c^2+d^2k}\right) + \left(\frac{bc-ad}{c^2+d^2k}\right)\sqrt{k}.$$

Note that $\mathbb{Q} \subset \mathbb{Q}(2) \subset$ the set of surds $\subset \mathbb{R}$.

Theorem 4. For F(k) as above, suppose the coefficients of $x^3 + ax^2 + bx + c = 0$ are all in F and that $r + s\sqrt{k}$, an element of F(k), is a root. Then some element of F is a root.

Proof. We may assume that $s \neq 0$, since otherwise we're done.

We have $0 = (r + s\sqrt{k})^3 + a(r + s\sqrt{k})^2 + b(r + s\sqrt{k}) + c = (r^3 + 3rs^2k + ar^2 + as^2k + br + c) + (3r^2s + s^3k + 2ars + bs)\sqrt{k}$. Write this as $A + B\sqrt{k} = 0$. So $A = B = 0$. Putting $r - s\sqrt{k}$ into the polynomial gives us $A - B\sqrt{k} = 0$, since only even powers of s occur in A and odd powers occur in every term of B. So $r - s\sqrt{k}$ is another root.

Now $x^3 + ax^2 + bx + c = (x - x_1)(x - x_2)(x - x_3)$
$= x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3$,
where x_1, x_2, and x_3 are the roots. So let's take $x_1 = r + s\sqrt{k}$, $x_2 = r - s\sqrt{k}$. Then $a = -(x_1 + x_2 + x_3) = -(r + s\sqrt{k} + r - s\sqrt{k} + x_3) = -(2r + x_3)$, so $x_3 = -a - 2r$, an element of F. □

Theorem 5 (Main Theorem) Given cubic equation $x^3 + ax^2 + bx + c = 0$, where the coefficients are rational. If the equation has a surd as a root, then it has a rational root.

Proof. Suppose x_1 is a surd and a root. As a surd, x_1 is in some subfield $(\ldots(\mathbb{Q}(k_1))(k_2)\ldots)(k_n)$. To see this, start to calculate x_1 from 0 and 1. (Recall that by definition, a surd can be calculated from 0 and 1 by additions, subtractions, multiplications, divisions, and abstractions of square roots.) Let $\sqrt{k_1}$ be the first non-rational square root we extract. Continue, until we must extract a square root, $\sqrt{k_2}$, not in $\mathbb{Q}(k_1)$. Continuing in this fashion, we get the above subfield.

By Theorem 4, the given cubic equation has a root in $(\ldots(\mathbb{Q}(k_1))(k_2)\ldots)(k_{n-1})$. Applying Theorem 4 a total of n times, we see that the cubic equation has a root in $\mathbb{Q}$. □

5. NONCONSTRUCTABILITY PROOFS

Theorem 6. The cube cannot be duplicated. In other words, given the edge of a unit cube (a unit segment), we cannot construct (with compass and straightedge) the edge of a cube of twice the volume. (The edge of such a cube would be $\sqrt[3]{2}$.)

Proof. We can think of this as being given surd-points (0,0) and (1,0) and being asked to construct $(\sqrt[3]{2},0)$. So it suffices to show that $\sqrt[3]{2}$ is not a surd.

Suppose it were. Then the cubic equation $x^3 - 2 = 0$ has a surd as a root. By the Main Theorem it has a rational root. But by the Rational Root Test, the only possible rational roots are ± 1 and ± 2 which are not roots. So $\sqrt[3]{2}$ is not a surd. □

Theorem 7. There are angles that cannot be trisected with compass and straightedge.

Proof. We actually show that no 60° angle can be trisected. Given a 60° angle, we can choose a coordinate system so that $A = (1, \sqrt{3})$, $B = (0,0)$, $C = (2,0)$ and the given angle is $\angle ABC$. (Note that A,B, and C are surd-points

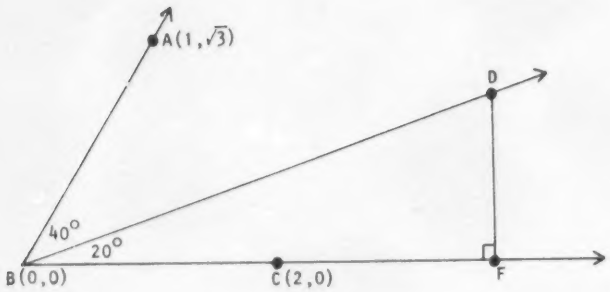

Figure 3.

which form the vertices of an equilateral triangle.) We want to show that there is no surd-point D such that the measure of angle DBC = 20°.

Suppose there were such a D. Let $\overline{DF}$ be the perpendicular to the x-axis. F is a surd-point since it lies on the two surd-curves y = 0 and x = (x-coordinate of D). Since the distance between two surd-points is a surd, cos 20° = BF/BD is a surd. Next we shall use the standard trigonometric identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(2A) = 2 \sin A \cos A$$
$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$1 = \sin^2 A + \cos^2 A$$

Now $\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta = (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta = (2\cos^2\theta - 1 - 2(1 - \cos^2\theta))\cos\theta = (4\cos^2\theta - 3)\cos\theta$. Since $\cos 60^{\circ} = 1/2$, we let $\theta = 20^{\circ}$ to see that $\cos 20^{\circ}$ is a solution of $1/2 = 4y^3 - 3y$. Letting $y = x/2$, the surd $2\cos 20^{\circ}$ is a root of $x^3 - 3x - 1 = 0$. By the Main Theorem, $x^3 - 3x - 1 = 0$ has a rational root. But the only possibilities are ± 1, which are not roots. This contradiction implies the Theorem.

Theorem 8. It is impossible to construct a regular seven sided polygon (heptagon) with compass and straightedge.

Proof. Suppose we could. Then we can construct the central angle, $\theta = 360^{\circ}/7$. And so, as before, $x_0 = \cos\theta$ is a surd.

Now $3\theta + 4\theta = 360^\circ$, so $\cos 3\theta = \cos(360^\circ - 4\theta) = \cos 4\theta$. So $4\cos^3\theta - 3\cos\theta = 2\cos^2 2\theta - 1 = 2(2\cos^2\theta - 1)^2 - 1$. Hence x_0 is a solution of $4y^3 - 3y = 2(2y^2 - 1)^2 - 1$, $4y^3 - 3y = 8y^4 - 8y^2 + 1$, $16y^4 - 8y^3 - 16y^2 + 6y + 2 = 0$.

So $2x_0$ is a root of $x^4 - x^3 - 4x^2 + 3x + 2 = 0$. Since 2 is a root of this, we see that the left hand side equals $(x - 2)(x^3 + x^2 - 2x - 1)$. But $x_0 = \cos\theta \neq 1$, so $2x_0 \neq 2$, and $2x_0$ is a surd and a root of $x^3 + x^2 - 2x - 1 = 0$. By the Main Theorem, there must be a rational root. But neither ± 1 are roots, so we have a contradiction. □

6. SUMMARY

First some algebraic background. Subfields of the real numbers are subsets of the real numbers that contain 0 and 1, and that are closed under (non-zero) division and subtraction. The Rational Root Test allows us to find all rational roots of a polynomial with integer coefficients.

Next we consider constructions. The subfield of surds is the smallest subfield in which we can take all square roots. The basic property of ruler and compass construction is that if we start with surd-points, then we can construct only surd-points.

In final preparation we need a basic algebraic result. We extend a subfield by including some square roots and the numbers needed to make our new set a subfield. For a cubic equation with rational coefficients, if there is a surd root (which is necessarily in some finite extension of the rationals) then there is a rational root.

Finally, we suppose we could trisect a 60° angle (defined using 3 surd-points) to get a 20° angle. Then,

since we can construct only surd-points, we show using standard trigonometric identities that the cubic equation $x^3-3x-1 = 0$ has a surd root. Hence, by the previous paragraph, it has a rational root. But that is contradicted by the Rational Root Test.

7. EXERCISES

SECTION 2

1. Prove that $\sqrt{3}$ is not rational.
2. Find all the roots of
   a. $2x^3 + x^2 - 5x + 2 = 0$
   b. $x^3 - 2x^2 + 1 = 0$
3. Prove that any subfield contains the set of rational numbers.
4. Prove that the set of all numbers of the form $a + b\sqrt{2}$, where a and b are rational, is a subfield.

SECTION 3

5. Complete the proof of Theorem 2. In other words, prove that P is a surd-point if P lies on two distinct surd-curves:
   a. which are lines;
   b. one of which is a line and the other a circle.
6. Show that the curves in Figure 2 are surd-curves.
7. Prove that the distance between two surd-points is a surd.

SECTION 4

8. Prove that $x^3 - x + 2 = 0$ has no surds as roots.
9. Prove that $x^3 - 2 = 0$ has no surds as roots.
10. Characterize all subfields of the form $\mathbb{Q}(k)$ as follows:
    a. Show that $\mathbb{Q}(p/q) = \mathbb{Q}(pq)$, for p and q positive integers.
    b. Show that $\mathbb{Q}(a^2p) = \mathbb{Q}(p)$, for a and p positive integers.
    c. Show that if p and q are positive integers greater than one, neither of which contains a perfect square (other than 1) as a factor, and $\mathbb{Q}(p) = \mathbb{Q}(q)$, then $p = q$.
    d. Conclude that we get a complete non-repititious list of subfields of the form $\mathbb{Q}(k)$ by letting k range over the integers greater than one which contain no non-trival squares as factors.

11. Show that $(\mathbb{Q}(2))(3)$ is not equal to any $\mathbb{Q}(k)$.

SECTION 5

12. Assume without proof that π is not a surd. Then prove that, in general,
    a. We cannot construct a segment whose length is the circumference of a circle with given diameter segment.
    b. We cannot construct a square whose area equals the area inside a circle, with a given diameter segment (known as squaring the circle).

    (Hint: Given a segment and a ray, it is possible to construct a point on the ray whose distance from the endpoint of the ray is the length of the segment.)
13. Develop a scheme which will trisect any given angle if its measure is $p \cdot 90^\circ/2^n$ where p and n are integers. (Hint: a) It's possible to tell whether angles coincide. b) It's possible to construct an angle with measure 60°. c) It's possible to bisect any angle. d) It's possible to 'copy' an angle in another location.)
14. Prove that it is impossible to trisect an angle of 30° with a compass and straightedge.
15. Construct a regular n-sided polygon with compass and straightedge for n = 3,4,6,8.

8. SOLUTIONS TO MOST EXERCISES

1. Mimic the proof of the Aside in Section 1. Suppose $3 = p/q$, in reduced form. Then $p^2 = 3q^2$, so 3 divides p^2, and hence 3 divides p. So $p = 3m$ for some integer m. Then $9m^2 = 3q^2$, or $q^2 = 3m^2$, and 3 divides q^2, so 3 divides q. But then p/q is not in reduced form.

2. a. By the Rational Root Test, the only possible rational roots are ± 1, ± 2, $\pm 1/2$. Since 1, -2, 1/2 work, they must be all three roots.

   b. As in a., we only need check ± 1. Since 1 is a root, we can factor: $x^3 - 2x^2 + 1 = (x - 1)(x^2 - x - 1)$. By the quadratic formula, the other two roots are $(1 \pm \sqrt{5})/2$.

3. Since any subfield, F, contains 0 and 1 and is closed under subtraction, it contains $-1 = 0 - 1$. Let n be an integer of smallest magnitude not in F, so $n - (\pm 1)$ is in F, and hence $n = n - (\pm 1) - (\mp 1)$ is in F, a contradiction. Hence F contains all integers. Since F is closed under non-zero division, it contains all rational numbers, p/q.

4. One must check that we have closure under subtraction and non-zero division (by rationalizing the denominator). This is done in Section 4 (take k = 2).

5. a. Suppose P lies on $Ax + By + E = 0$ and $Cx + Dy + F = 0$ with all coefficients surds. Since these are distinct lines which meet, the difference of their slopes, $(-A/B) - (-C/D) = -(AD - BC)/BD$, is non-zero. So for $d = AD - BC$, d is a non-zero surd. (If one of the lines has infinite slope, then either $d = -BC \neq 0$ or $d = AD \neq 0$, and d is still a non-zero surd.) Solving, we get $x = (BF - DE)/d$ and $y = (CE - AF)/d$, both surds.

   b. Suppose P lies on $Jx + Ky + L = 0$ and $x^2 + y^2 + Dx + Ey + F = 0$, with all coefficients surds. Proceed exactly as in the last paragraph of the proof of Theorem 2.

6. The circle has equation $x^2 + y^2 + (-1) = 0$ and the line equation $-1(x) + 4y + (-2) = 0$.

7. If (x_1,y_1) and (x_2,y_2) are surd points, then x_1, x_2, y_1, y_2 are all surds. So the distance between the points,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ ,\ \text{is a surd.}$$

8. By the Rational Root Test the only possible rational roots are ± 1 and ± 2. These are not roots, so by the Main Theorem there are no surd points.

9. Same argument as 8.

10. a. $\mathbb{Q}(p/q)$ consists of all numbers of the form $a + b\sqrt{p/q} = a + (b/q)\sqrt{pq}$ where a and b are in $\mathbb{Q}$. But this gives us all the elements of $\mathbb{Q}(pq)$.

    b. An element of $\mathbb{Q}(a^2p)$ is the form $c + d\sqrt{a^2p} = c + da\sqrt{p}$. This is an element of $\mathbb{Q}(p)$ and every element of $\mathbb{Q}(p)$ can be written this way.

    c. Since $\mathbb{Q}(p) = \mathbb{Q}(q)$, $\sqrt{p}$ is in $\mathbb{Q}(q)$, so $\sqrt{p} = a + b\sqrt{q}$ for some a and b in $\mathbb{Q}$. Squaring, we get $p = (a^2 + b^2q) + 2ab\sqrt{q}$. Since every element of $\mathbb{Q}(q)$ has a unique representation in the standard form, $ab = 0$. So $a = 0$ or $b = 0$. If $b = 0$, then $p = a^2$, contradicting the assumption that p has no non-trivial square factor (a must be an integer, since p is, and $a \in \mathbb{Q}$). So we must have $a = 0$, and $p = b^2q$. Write $b = k/m$ in reduced form. Then $m^2p = k^2q$. Since p and q have no non-trivial square factors, $m = k = 1$ and $p = q$.

    d. From a and b we see that we get all such subfields, and c tells us that our list doesn't repeat.

11. By 10, if $(\mathbb{Q}(2))(3) = \mathbb{Q}(k)$ for some k, we may assume k is a positive integer greater than one with no non-trivial squares as factors. But $\sqrt{2} \in \mathbb{Q}(k)$ and $\sqrt{3} \in \mathbb{Q}(k)$. By the proof used in 10 c., we get $2 = k = 3$, an impossibility.

12. a. Choose a coordinate system so that the ends of the given diameter have coordinates (0,0) and (1,0). If we could construct a segment, with length the circumference, we could construct (with the hint) the point $(\pi,0)$. Since $(\pi,0)$ is not a surd-point, this contradicts Theorem 3.

b. As in a., we could construct the point $(\sqrt{\pi}/2),0)$. This is not a surd-point, since $(\sqrt{\pi}/2)(\sqrt{\pi}/2)4 = \pi$, so we have a contradiction.

13. Start to list the pairs (p,n), by listing the pairs whose absolute values of coordinates add to 0,1,2,...,((0,0), (1,0), (0,1), (-1,0), (0,-1), (1,1), (2,0),...). For each pair, (p,n) construct a 60° angle. Bisect (or double if negative) n + 1 times to get a $30^{\circ}/2^{n}$ angle. Copy this angle |p| times to get a $|p|\ 30^{\circ}/2^{n}$ angle. Copy this angle, α, 3 times and see whether it can be placed to coincide with the given angle. If so, α can be placed to determine the angle trisector.

14. If we could trisect a 30° angle, then we could take a 60° angle, bisect it to get a 30° angle, trisect that to get a 10° angle, double that to get a 20° angle (see 13) and we would have trisected a 60° angle. This is impossible by the proof of Theorem 7.

UMAP

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

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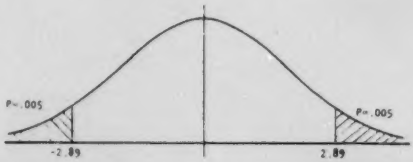
MODULE 268

Testing a Hypothesis: t-Test for Independent Samples

by Herbert L. Kayne



Applications of Statistics

TESTING A HYPOTHESIS:

t-TEST FOR INDEPENDENT SAMPLES

by

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Intermodular Description Sheet: UMAP Unit 268

Title: TESTING A HYPOTHESIS: t-TEST FOR INDEPENDENT SAMPLES

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Review Stage/Date: IV 4/29/80

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Suggested Support Material: Hand calculator, 't' table, and references to source material.

References:

Bryant, E.C. (1966). Statistical Analysis, 2nd ed. McGraw-Hill, Inc., N.Y.

Dixon, W.J. and F.J. Massey (1969). Introduction to Statistical Analysis. McGraw-Hill, Inc., N.Y.

Hoel, P.G. (1965). Introduction to Mathematical Statistics, 3rd ed. John Wiley and Sons, Inc., N.Y.

Snedecor, G.W. and W.G. Cochran (1967). Statistical Methods, 6th ed. Iowa State University Press, Ames.

Prerequisite Skills:

1. Define: frequency distribution, Gaussian ('normal') distribution, t-distribution for small samples, mean, variance, degrees of freedom, standard deviation and standard error.
2. Given a small sample of measurement data (continuous variable), calculate the mean, variance, standard deviation and standard error.

Output Skills:

1. Perform the t-test on appropriate experimental data and interpret the calculated value of 't'.
2. Understand the role of a statistical test of significance in the research process. Recognize the relationship between the statistical arithmetic and the design and conduct of the experimental investigation.

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1. THE BIOLOGICAL PROBLEM

In evaluating a biological experiment, an investigator often wants to know whether the results obtained under the experimental condition are really different from the results under the control condition. The following statistical test, the t-test for independent groups, allows the experimenter to answer that question with confidence. The experimental procedure, the rationale of the statistical test, the analysis of the data and the interpretation will be presented. This is an application of statistics to biological research as well as to research in several other disciplines.

2. A COMPARATIVE EXPERIMENT

2.1 Setting up the Experiment

Based on prior knowledge and reasoning, an investigator has an idea, for example, that a particular hormone affects the calcium ion concentration of heart muscle. That is, the calcium ion concentration of a group of animals injected with the hormone will differ from the calcium ion concentration in a group of animals not given the hormone. Assume there are 20 animals available for the experiment. By means of a coin flip, each animal is "randomly" allocated to either the experimental group or the control group. The result might be 9 animals in the experimental group and 11 in the control group. The animals are treated for one week. On each day each animal is given an injection -- the experimental animals get the hormone and the control animals are injected with the solvent in which the hormone is dissolved. At the end of the treatment period, the animals are sacrificed and the calcium ion concentration is determined for each heart.

2.2 Features of the Chosen Design

There are two critical features of the design and conduct of this experiment. First, by random allocation of animals to groups, one tries to ensure that there are no systematic differences between the two groups prior to the actual experiment. One assigns to "chance" the task of making the groups comparable. This will be the basis of the statistical arithmetic. Second, the comparison is made with all animals in both groups treated as nearly alike as possible. Thus, it is assumed that the only difference between the two groups is the presence or absence of the hormone. The conclusion depends on this.

An incidental feature is that the sample sizes need not be equal. Under most circumstances, samples of equal size do represent the most efficient utilization of experimental material, but minor inequality does not present a serious problem. In the above example, were the allocation to be rather extreme, e.g., 13 and 7, one would simply discard it and try again.

3. EXPERIMENTAL DATA

TABLE I

Calcium Ion Concentration of Heart Muscle

(micrograms per gram of wet weight)

Control Group (n = 11)	Experimental Group (n = 9)
189	122
172	215
154	206
230	159
193	230
110	211
134	241
174	190
173	199
192	
160	

Symbolically, each value is denoted as X_i, the sample size as n, the sample mean as $\bar{X}$, and the sample variance as s^2. Where

$$\bar{X} = \frac{\Sigma X_i}{n}$$ [1]

and

$$s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n - 1}.$$

4. STATISTICAL RATIONALE

4.1 The Null Hypothesis

Prior to making the calcium measurements, tentatively assume that in terms of calcium ion concentration of heart muscle the two groups of animals are random samples from one population. In common usage, this is expressed in several ways: 1) there is no real difference between the two groups, 2) the mean calcium ion concentration of the control group is equal to the mean of the experimental group, 3) the hormone had no effect on calcium ion concentration, or 4) whatever variation there is among the 20 measured values of calcium is due to sampling variation from one population. Specifically, this tentative assumption is called the null (no difference) hypothesis. Symbolically the *null hypothesis* for the control and experimental samples is denoted,

$$\bar{X}_{control} - \bar{X}_{experimental} = 0.$$

4.2 Test of Significance

Having made the experimental measurements, a statistical test of significance is performed. The test asks, "What is the probability that chance alone is responsible for the discrepancy between the experimental result and the null hypothesis?" If this probability is large, the

---

[1] In statistics, summations are employed so much that it has become a convention to use Σ instead of the more precise notation $\sum_{i=1}^{n}$. In keeping with this convention "Σ" is used for "$\sum_{i=1}^{n}$" throughout this module.

null hypothesis will be accepted. If the probability is small, the null hypothesis will be rejected. Notice two things. The answer to the test question is not an absolute one; it is a probability statement. Moreover, acceptance or rejection of the null hypothesis requires that an arbitrary decision be made as to what constitutes "large" and "small."

A test of significance is a ratio:

$$\frac{\text{a measure of the effect}}{\text{a measure of the variation}}.$$

In effect it asks, "How big is the difference between the two means relative to the uncertainty associated with that difference?" From the context of this experimental situation, it is reasonable that for this so-called t-ratio the numerator is the difference between the mean of the experimental group and the mean of the control group. What is no apparent is the nature of the denominator. The denominator reflects the variation one can expect between two means by random sampling from one population.

The following statements describe what the denominator is. For a rigorous development of why it is that, one should consult a textbook of mathematical statistics.[2] Assume that calcium ion concentration of heart muscle is a continuous variable that tends to follow the Gaussian distribution.[3] The variance (s^2) of a sample is the estimate of theoretical population variance, σ^2. The variance of a sample mean is estimated by $\frac{s^2}{n}$. The variance of the difference between two means is equal to the sum of the variance of the two means -- the denominator -- is equal to the square root of the variance of the difference.[4]

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[2] Hoel, P.G., Introduction to Mathematical Statistics, 3rd ed., John Wiley & Sons, Inc.

[3] For large samples, the assumption of Gaussian distribution is not necessary; for small samples in which this assumption is unlikely to hold, there are alternatives to the t-test.

[4] For a detailed analysis of testing a hypothesis see [S-1].

4.3 The t Statistic (Ratio)

Were the sample sizes equal, the t-ratio would then be:

$$t = \frac{\bar{X}_C - \bar{X}_E}{\sqrt{\frac{s_C^2}{n_C} + \frac{s_E^2}{n_E}}}.$$

When the sample sizes differ, as in our example, a weighted average of s_C^2 and s_E^2 is used. Weight is based on the number of data in each sample. The degrees of freedom are respectively, $n_C - 1$ and $n_E - 1$. The resulting ratio is:

$$t = \frac{\bar{X}_C - \bar{X}_E}{\sqrt{\frac{\Sigma(X_C - \bar{X}_C)^2 + (X_E - \bar{X}_E)^2}{n_C + n_E - 2}\left(\frac{1}{n_C} + \frac{1}{n_E}\right)}}.$$

As an exercise in algebra, it is left for the curious reader to show that this expression reduces to the expression above when $n_C = n_E$. (Recall that $s^2 = \Sigma(X_i - \bar{X})^2/n - 1$.)

5. DATA ANALYSIS

Control Group	Experimental Group
$\bar{X} = 171.00$	$\bar{X} = 208.11$
$n = 11$	$n = 9$
$\Sigma(X_i - \bar{X})^2 = 10,244.0$	$\Sigma(X_i - \bar{X})^2 = 4,636.88$

$$t = \frac{171.00 - 208.11}{\sqrt{\frac{10,244.0 + 4,636.88}{18}\left(\frac{1}{11} + \frac{1}{9}\right)}}$$

$$t = -2.87.$$

6. INTERPRETATION

6.1 Importance of Degrees of Freedom

The original question was, "What is the probability that chance alone is responsible for the discrepancy between the experimental result and the null hypothesis?" Now that question becomes, "What is the probability of getting a t-value as large as -2.87 by random sampling?"

The t-distribution is a theoretical probability distribution that is symmetrical and bell-shaped, like the Gaussian curve. In addition, there is a different t-distribution for each *degree of freedom*. (For degrees of freedom above 30, the t-distribution is very similar to the Gaussian distribution; in fact, we can say that the Gaussian distribution is t with infinite degrees of freedom.) The degrees of freedom for the t-test for independent groups is defined as $(n_C - 1) + (n_E - 1)$. The t-values and their corresponding probabilities are tabulated in most statistics texts.[5] Part of a t-table is shown here.

Degrees of Freedom	Probability (two-tailed) .10	.05	.01	.001
5	2.02	2.57	4.03	6.86
10	1.81	2.23	3.17	4.59
18	1.73	2.10	2.89	3.92
30	1.70	2.04	2.75	3.65

For 18 degrees of freedom, the t-value of 2.87 has a probability of about .01.

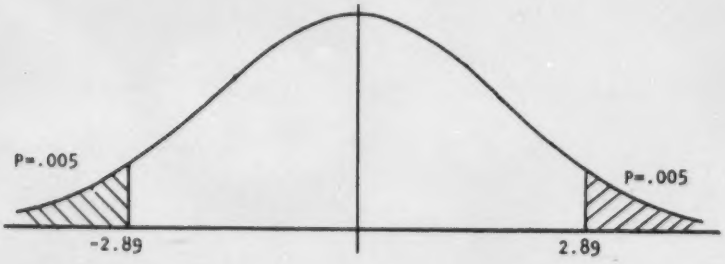
6.2 The Level of Significance

Most biological investigators agree that a probability of .05 or less is "small." Thus, in keeping with the aforementioned ground rules, the null hypothesis is rejected in this example. It is improbable that sampling

[5] Snedecor, G.W. and W.G. Cochran, Statistical Method, 6th ed., Iowa State University Press, Ames.

variation alone is responsible for the experimental results. The two groups differ by more than one would expect by chance, and since the groups are comparable except for the presence or absence of the hormone, the conclusion is that the hormone altered the calcium ion concentration.

Since the alternate hypothesis did not specify the direction of the difference, the two-tailed probability was used for the t-value of -2.87. The tabulated t-distribution for df = 18 shown graphically is:



If the investigator had specified, a priori, that if there were a difference the experimental mean would be larger than the control (as it turned out), then the one-tailed probability associated with the calculated t-value of -2.87 would be .005.

7. CONFIDENCE IN THE CONCLUSION

The statistical t-test is designed to aid the investigator in making a decision. Where treatment effects are small (but perhaps important) and biological variation among individuals is large, the test can be particularly helpful. However, having rejected the null hypothesis in this example, the question remains, "How confident can the investigator be?" At least three aspects of the experiment must be considered -- the allocation of animals to the two

groups, the physical conduct of the experiment, and the statistical procedure.

One relies on random allocation to control all extraneous factors. Randomization holds in the long run; but in the short run, as one would expect, it might not do its job completely. Short of checking one or two factors, such as body weight of the animals in this example, the investigator cannot evaluate the vicissitudes of randomization. Whether all factors pertinent to calcium ion concentration of heart muscle are balanced among the two groups remains unknown.

It is in the actual conduct of the experiment that factors other than the presence or absence of hormone are most likely to bias the outcome. It is difficult for the investigator and other participants in the experiment to treat the two groups of animals exactly the same. An essential safeguard is to keep the participants "blind" to the group designation of each animal. This minimizes their subconscious tendency to bias the outcome. Unfortunately some treatments defy masking. In general, the magnitude of investigator bias cannot be evaluated.

The uncertainty in the statistical test can be quantified. By arbitrarily selecting a probability of .05 or less for rejecting the null hypothesis, one knows that the chance of rejecting a true null hypothesis is .051. In this example, t-values of 2.87 or greater can occur by random sampling from one population; they will occur almost 1 in 100 times. According to the ground rules of the test of significance, the null hypothesis will be rejected every time the t-value is $\geq$ 2.87. One out of a hundred times will be an error.

As part of the experimental plan, the investigator can control this error by altering the critical probability. For example, by defining "small" as a probability of .0001 or less, one minimizes the probability of erroneously

rejecting the null hypothesis. But by so doing, one maximizes the probability of accepting a null hypothesis that is actually false and should be rejected.

So while the intent of the statistical test was to increase confidence in the conclusion, it is nonetheless true that uncertainty still remains. Hence the word, "research." Only by replication of the experiment, particularly by other investigators and in other settings, is one's confidence fortified.

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8. EXERCISES

1. An experiment was designed to test the effect of a vitamin supplement on weight gain in mice. Animals were randomly allocated to two groups. One group was given ordinary chow and the other was given chow to which the vitamin supplement had been added. Each mouse was weighed at the beginning of the experiment and after 10 days. The data are expressed as gain in body weight (grams) in 10 days.

Chow	Chow + Vitamin
4.0	6.4
5.7	4.3
4.2	5.5
5.3	4.5
5.1	

   Does the vitamin supplement alter growth?

2. The effect of ambient temperature on food intake was studied in rats. One group of 10 rats was maintained at 25°C. Their average food intake over a period of 21 days was 161 grams with a standard deviation of 9 grams. The other group of 10, maintained at 20°C, had a mean intake of 204 grams with S.D. = 12. Is there a statistically significant difference between the two groups?

3. One of the important questions that an investigator must answer before he begins an experiment is, "How many animals shall I use?" Consideration of the t-ratio allows one to make an educated guess at the answer. Basically one uses preliminary data and/or makes reasonable estimates of all the terms in the t-ratio except 'n' and then solves for 'n'. For example, an investigator was planning an experiment to measure the effect of removal of the testes on developed tension of heart muscle in dogs. In a pilot study on several normal dog hearts, the mean developed tension was 2.4 grams with a standard deviation of .4 grams. If removal of the testes were to have an appreciable effect, the investigator guessed that the mean tension might be reduced by .5 grams. How many animals should be used in the control and experimental groups in order to detect a significant difference at a probability of .05? 10

9. ANSWERS TO EXERCISES

1.

Chow	Chow & Vitamins
$\Sigma X = 24.30$	$\Sigma X = 20.70$
$n = 5$	$n = 4$
$\bar{X} = 4.86$	$\bar{X} = 5.18$
$\Sigma(X_i - \bar{X})^2 = 2.13$	$\Sigma(X_i - \bar{X})^2 = 2.83$

$$t = \frac{\bar{X}_C - \bar{X}_{CV}}{\sqrt{\frac{\Sigma(X_C - \bar{X}_C)^2 + \Sigma(X_{CV} - \bar{X}_{CV})^2}{n_C + n_{CV} - 2}\left(\frac{1}{n_C} + \frac{1}{n_{CV}}\right)}}$$

$$t = \frac{4.86 - 5.18}{\sqrt{\frac{2.13 + 2.83}{7}\left(\frac{1}{5} + \frac{1}{4}\right)}} = -.56$$

$df = (5 - 1) + (4 - 1) = 7.$

By inspection of the abbreviated t-table in this unit, $P > .05$. Accept the null hypothesis; there is insufficient evidence to conclude that the vitamin supplement alters growth. Notice the implication -- *if* there were data on more animals there *might* be evidence to reject the null hypothesis. Accepting the null hypothesis is not the same as 'proving' that the two groups are the same.

2.

20^oC	25^oC
$\bar{X} = 204$	$\bar{X} = 161$
$n = 10$	$n = 10$
S.D. = 12	S.D. = 9
$s^2 = 144$	$s^2 = 81$

$$t = \frac{\bar{X}_{20} - \bar{X}_{25}}{\sqrt{\frac{s^2_{20}}{n_{20}} + \frac{s^2_{25}}{n_{25}}}} = \frac{204 - 161}{\sqrt{\frac{144}{10} + \frac{81}{10}}} = 9.06$$

df = 18, $P < .001$, reject null hypothesis.

3. Assume that the two means will be 2.4 and 1.9 (1.9 is a reduction of .5 from the mean of 2.4 in the pilot study). Assume that the standard deviations will both be .4 (as was found in the pilot study) so that the variances, s^2 will both be .16. Since, at this point, the degrees of freedom are unknown, assume that the t-value at $P = .05$ is 2.0. In the expression for 't', solve for n:

$$2 = \frac{2.4 - 1.9}{\sqrt{\frac{.16}{n} + \frac{.16}{n}}}$$

n = 5, roughly.

Were there to be 5 animals in each group, the degrees of freedom would be 8. The 5% value of t at df = 8 is 2.3, so sample sizes of 6 or 7 would be safer. How conservative the investigator guesses relates to the importance associated with a difference between means of a given amount as well as to the resources available for the experiment. At best, guess work can only result in a "ball-park" estimate.

10. MODEL EXAM

1. a. Cerebral blood flow was measured in 10 dogs under anesthesia. Hemorrahagic shock was induced in 5 dogs by removing 20% of their circulating blood volume. The other 5 dogs were the controls. All blood flow measurements were made one hour after induction of anesthesia, and this was also 30 minutes after hemorrhage in the experimental animals. The flow measurements (in ml/min/100 g brain tissue) were:

Control	Experimental
10.7	9.5
12.1	7.1
11.9	8.1
8.6	6.2
10.0	7.0

Does the data imply that there is a significant difference in blood flow to the brain following severe hemorrhage?

b. Could this experiment have been designed in another way that might be more sensitive in detecting the effect of hemmorhage on blood flow?

2. A neurobiologist was studying the incorporation of amino acid into protein in the brain of rats at 15 days of age. She kept one group of 6 rats in a single cage. Another group of 6 rats were kept in individual cages. These animals tend to be more aggressive than those living together. After a 'treatment' period of 2 weeks, radioactive amino acid was injected intravenously into each animal; one hour later the animals were sacrificed and the brain was dissected out for measurement of radioactivity. For each of the two groups, the data are presented in summary form (mean and standard deviation) rather than the uptake of radioactivity for each animal.

	Specific radioactivity (counts/min/mg of tissue)
Single cage	3600 ± 400 (mean ± S.D. in 6 rats)
Individual cages	2300 ± 500 (mean ± S.D. in 5 rats; 1 died)

Does it seem as if the environment has altered the uptake of amino acid?

3. An ornithologist at University A has been studying the body temperature of a particular species of bird that he trapped in the wild. A colleague at near-by University B had trapped this species 2 years before, and he too had recorded body temperature as part of a more complicated study. He decided to compare the body temperature data via t-test, and he was surprised to find a striking difference ($P < .001$) between the data of ornithologists A and B. Why should one not be surprised?

11. ANSWERS TO MODEL EXAM

1. a. $t = 3.59$, $P < .05$, reject the null hypothesis; hemorrhage decreases blood flow to the brain.

   b. Yes. A more sensitive way would be to make measurements of flow before shock and after shock in each animal. Each animal then serves as its own control. The t-test must be modified and is known as a paired t-test. In part (a), technical reasons negated the measurement of brain blood flow twice in the same animal; hence the design involves two independent groups.

2. For the 'single cage' group:

   s(or SD) = 400; therefore $s^2 = 160{,}000$.

   $$s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n - 1}; \quad 160{,}000 = \frac{\Sigma(X_i - \bar{X})^2}{5};$$

   $$\Sigma(X_i - \bar{X})^2 = 800{,}000.$$

   For the 'individual cages' group, the above calculations result in $\Sigma(X_i - \bar{X})^2 = 1{,}000{,}000$. Finally,

   $$t = \frac{3600 - 2300}{\sqrt{\frac{800{,}000 + 1{,}000{,}000}{9}\left(\frac{1}{6} + \frac{1}{5}\right)}}$$

   $t = 4.80$ $(P < .05)$.

   Reject the null hypothesis; animals in individual cages have reduced uptake of amino acid.

3. Any number of factors could be responsible for this result. Ornithologist B might have trapped a different subset of the wild-ranging population. His thermometer might have differed systematically from that of ornithologist A. The ambient conditions might have differed between the two times and in some way influenced body temperature, etc., etc. The fact is that when one can neither employ random allocation nor make measurements under comparable conditions, a variety of explanations can account for an apparent difference. Plugging numbers into a t formula is unwise unless the experimental design warrants it.

However, there are circumstances where one *cannot* randomly allocate individuals to groups. This is particularly true when one wants to compare normal (presumably healthy) human subjects with patients who have a specific disease. One must be extremely cautious in interpreting a test of significance in this case.

SPECIAL ASSISTANCE SUPPLEMENT

[S-1]

We can formalize the procedure for testing a hypothesis by considering seven steps.

1. State the null hypothesis H_o and the alternative H_1.
2. Choose a level of significance. This is usually referred to as 'α'. The choice of α is arbitrary; however, it is standard to use $\alpha = .05$ or $\alpha = .01$. Please consult a statistics text for a detailed discussion of Type I and Type II errors.
3. Decide on the distribution and the statistic that will best analyze your problem. In this case we use the t-distribution, and the statistic is:

$$t = \frac{\bar{X}_C - \bar{X}_E}{\sqrt{\frac{\Sigma(X_C - \bar{X}_C)^2 + \Sigma(X_E - \bar{X}_E)^2}{n_C + n_E - 2}\left(\frac{1}{n_C} + \frac{1}{n_E}\right)}}.$$

It is important to realize that this is not the only way t can be expressed. Again, you may wish to consult a statistics text for other motivations for and forms of the t-distribution.

4. Choose a region of rejection. In the two-tailed example that is employed in this unit we have:

$t \leq -2.10$ or $t \geq 2.10$

assuming $\alpha = .05$. For a single (upper) tail test we have $t \geq 1.73$ for $\alpha = .05$.
If $\alpha = .01$ we have $t \leq -2.89$ or $t \geq 2.89$ for a two-tailed test, and $t \geq 2.55$* for an upper tail test. All of these statements are of course based on 18 degrees of freedom.

5. Computations: Do the proper substitutions to get a calculated t value as presented in Section 5.

*See Bryant E.C., Statistical Analysis, 2nd ed., McGraw-Hill, Inc. for a complete t Table.

6. Reject or do not reject the hypothesis depending on whether the calculated t is in or out of the stated region of rejection. (If the t is in the region of rejection, we reject the hypothesis.)

   (Note: A *statistician* would probably not say that we accept an hypothesis no matter how heavy the weight of the evidence. We must keep in mind that we are examining a sample. Even though our analysis of a sample gives us no mathematical grounds to reject, we cannot be sure that an alternative is in fact true. In short, it is better to say "reject H_o" if in essence we believe H_1 should be accepted, and say reject H_1 if we believe we should accept H_o.)

   This sixth step is called the statistical decision. In many cases we may still not reject an hypothesis even if the data and statistics suggest so. This brings us to step 7.

7. Make the scientific or management decision. Decisions of this kind are based on experience and other factors outside the experimental design. This is what is being discussed in Section 7.

[S-2]

The t statistic is sometimes written with the weighted average of s_C and s_E displayed explicity, i.e.,

$$t = \frac{\bar{X}_C - \bar{X}_E}{s\sqrt{(1/n_C) + (1/n_E)}}, \text{ where } s^2 = \frac{(n_C - 1)s_C + (n_E - 1)s_E}{(n_C - 1) + (n_E - 1)}.$$

s^2 can be further simplified so that we can write:

$$s^2 = \frac{(n_C - 1)s_C + (n_E - 1)s_E}{n_C + n_E - 2}$$

The degrees of freedom are given by the demoninator n_C+n_E-2.

[S-3]

This supplement is here so that students appreciate the fact that there are some other statistical assumptions that must be made if we desire more precision.

The t- formula suggested here should only be used if the variances are equal ($\sigma_1^2 = \sigma_2^2$). To do things properly a test using $F = s_1^2/s_2^2$ should be done first. (Please refer to a statistics text for further details on the F- distribution.) If we do not reject H_o ($\sigma_2^2 = \sigma_2^2$), then it is all right to proceed with a test for the means using the t statistic suggested. If we are forced to reject H_o and assume $\sigma_1^2 \neq \sigma_2^2$ then a *common* approximation for the t- statistic, which is basically the same form as in the module, is used. However the degrees of freedom is calculated by a long involved expression. The t statistic and the degrees of freedom are:

$$t = \frac{\bar{y}_1 - \bar{y}_2 - (u_1 - u_2)}{\sqrt{s^2\bar{y}_1 + s^2\bar{y}_2}}$$

with $\dfrac{(s^2\bar{y}_1 + s^2\bar{y}_1)^2}{\left[(s^2\bar{y}_1)^2/(n_1+1)\right] + \left[(s^2\bar{y}_2)^2/(n_2+1)\right]} - 2$ degrees of freedom

where $s^2\bar{y}_1$ is the variance of the first sample mean and $s^2\bar{y}_2$ is the variance of the second sample mean.

We will not, with the expression above, get an integer in every instance for the degrees of freedom. However, one may obtain a value for the region of rejection by interpolating in the t table.

As it turns out, the example used does provide variances that are not significantly different. This, however, is only seen after a test of the hypothesis ($\sigma_1^2 = \sigma_2^2$) is done.

A test for equality of variances is sometimes called a test of homogeneity. We give that best below.

The Test of Homogeneity of Variances:

$s_1^2 = 1024.4$ and $s_2^2 = 579.61$ from the data.

1) H_o: $\sigma_1^2 = \sigma_2^2$ $\quad H$: $\sigma_1^2 > \sigma_2^2$ or $\sigma_1^2 < \sigma_2^2$
2) $\alpha = 0.10$ (arbitrary)
3) $F = s_1^2/s_2^2$ with $n_1 - 1$ and $n_2 - 1$ df.
4) Region of rejection: $F \geq 3.35$ with 10 and 8 degrees of freedom.*
5) $F = \frac{1024.4}{579.61} = 1.7667$
6) Do not reject H_o.

Alternatively, one could use Bartlett's test:

Suppose there are K(2 in this case) variances to be compared, denoted s_1^2 and s_2^2 with $n_1 - 1$ and $n_2 - 1$ degrees of freedom. Then the quality

$$\chi^2 = \frac{2.3026 \left[s_p^2 \Sigma \ (n_i - 1) - \Sigma \ (n_i - 1) \log s_i^2 \right]}{1 + \frac{1}{3(k-1)} \left[\Sigma \frac{1}{n_i - 1} - \frac{1}{\Sigma (n_i - 1)} \right]}$$

is distributed as χ^2 with K - 1 degrees of freedom. χ^2 calculated greater than the critical value for a specific α would suggest rejection of homogeneity i.e., $\sigma_1^2 \neq \sigma_2^2$. ($s_p^2$ is a pooled estimate of the variance.)

*See Dixon, W.J. & Massey, F.J. Introduction to Statistical Analysis. Table A-7 for complete set of F values.

UMAP

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS

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Α Β Γ Δ Ε Ζ Η Θ Ι Κ Λ Μ Ν Ξ Ο Π Ρ Σ Τ
α β γ δ ε ζ η θ ι κ λ μ ν ξ ο π ρ σ τ υ φ χ ψ ε ρ δ

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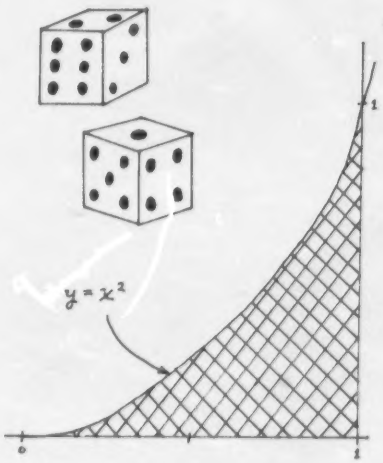
MODULE 269

Monte Carlo:

The Use of Random Digits to Simulate Experiments

Dale T. Hoffman



Applications of Probability

MONTE CARLO:

THE USE OF RANDOM DIGITS TO SIMULATE EXPERIMENTS

by

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Intermodular Description Sheet: UMAP Unit 269

Title: MONTE CARLO: THE USE OF RANDOM DIGITS TO SIMULATE EXPERIMENTS

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Review Stage/Date: IV 7/26/79

Classification: APPL PROBABILITY

Prerequisite Skills:
1. Know and understand the "relative frequency" definition of the probability of an event.
2. Construct a frequency histogram.
3. Calculate a mean, median, mode, and standard deviation.

Output Skills:
1. Use the Monte Carlo technique to simulate simple experiments.
2. Realize the strengths and weaknesses of this technique.
3. Better appreciate the role of approximate solutions to complex problems.

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1. PROBLEMS FOR SIMULATION

The world of science and business is full of mathematical problems which cannot be solved exactly or easily. But there is a general procedure which can be used to get workable answers to many of these problems. The following examples show the variety and complexity of the problems which the Monte Carlo technique can solve. They indicate the power and breadth of the application of this straightforward technique. Don't panic if the examples look difficult now, for some of them are. But after completing this module they should seem much easier.

1.1 The Rhythm Method

The rhythm method of birth control is known to be 70% effective. That is, the probability that someone, using this method by itself, will become pregnant in any one year is 30%. What is the expected number of years before someone who uses the method becomes pregnant?

1.2 Lottery

Each ticket in a lottery contains a single "hidden" letter. Among all the tickets, 50% contain a "W," 40% contain an "I," and 10% contain an "N." How many tickets should you expect to buy in order to be able to spell the word "WIN," and thus win a prize? To spell "IWIN?" (Variations of this spelling scheme are used by several state lotteries.)

1.3 Drunkard

A drunkard leans against a lamp post in the middle of a large plaza. He takes one step north or south or east or west and then stops. If he continues to step randomly (each direction is equally likely), how far would you expect him to be from the lamp post after 5 steps? After 10 steps? (A variation of this example is used to model the behavior of a molecule suspended in a liquid. The random motion exhibited by such molecules is called Brownian Motion, after the English botanist, Robert Brown. Brown reported in 1827 that an aqueous suspension of a pollen he was studying contained microscopic particles which carried out a continuous, haphazard zigzag movement. Brown was not the first to notice this phenomenon, but was the first to study it in detail, and was the first to notice that the movement could not be attributed to life in the particles themselves. For more about Brown, see the Encyclopaedia Britannica.)

1.4 Grocery Store

As the owner of a small grocery store you have a choice of hiring

(a) 2 cashiers who do their own bagging, and each of whom can check out a shopper in 2 minutes, or
(b) 1 cashier and 1 boxboy who, working as a team, can check out a shopper in 1 minute.

Based on your experience, you estimate that 30% of the time (minutes) no new shoppers get into the checkout line; 40% of the time, 1 new shopper gets into line; and 30% of the time, 2 new shoppers get into line. Using each checkout system, estimate

(1) the expected waiting time in line per shopper, and
(2) the expected line length a shopper will encounter.

Which checkout system would you adopt for your store? (This type of problem occurs frequently in business. For example, one could simulate the expected cost-performance of proposed inventory plans using data from previous years.)

Exercise 1

For each of the previous problems:
(1) Describe an actual experiment which could be used to obtain an approximate answer, and
(2) List some of the disadvantages of the direct experiments which you proposed in part (1).

---

2. INTRODUCTION

Frequently a scientist or someone in business wants to know how a "system" will behave. If the system is fairly simple, we can sometimes determine mathematically how it will behave. But as the system becomes more complicated, we may not know the necessary mathematics, or the equations may be too complex to solve, or the mathematics may not have been invented yet. At this point we could resort to a series of experiments: If we operated the system long enough, we could get a good idea about how it would behave under different circumstances. Unfortunately, this is not always practical or possible--the system may not have been built yet, the experiments may be very expensive in time or money, or be dangerous or immoral. In these cases we can sometimes run a series of simulated experiments--experiments which behave like the real thing, but which do not have the disadvantages of the

real experiments. If designed and run properly, these simulations will mimic the behavior of the real experiments and yield results quicker and cheaper.

In this module we will look at a particular type of discrete simulation called the Monte Carlo technique. It is very powerful and is widely used in science and business. It is attractive because it is easy to use, inexpensive in time and money, and because computers can perform much of the work.

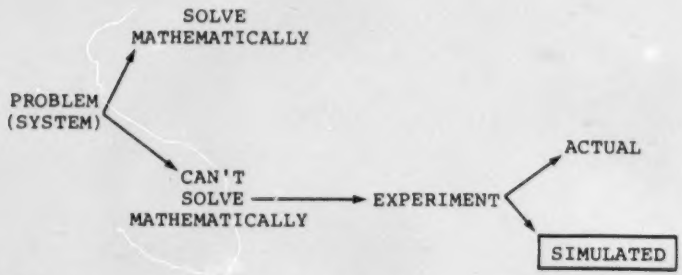

Figure 1. To solve a problem that we cannot solve mathematically, we may be led to simulate an experiment.

3. A WORKED EXAMPLE

3.1 Coin

How many times should we expect to flip a coin before we accumulate three heads?

The actual experiment in this problem would be to flip a coin until a total of 3 heads have appeared, and then to note how many flips were required. This single number, the number of flips, will be an approximation of the answer, but perhaps a poor approximation. To increase your confidence in the accuracy of your approximation, you could repeat the experiment many times, keeping a record of the outcomes.

NOTATION: Pr(A) = probability that event A occurs.

To save time and wear on your thumb, you could use random digits instead of a coin. On each flip there are only two possible outcomes, heads (H) or tails (T), each of which occurs with probability 0.5. If we let the occurrence of an even digit represent H and an odd digit represent T, then a sequence of digits, say 72362, would represent a sequence of outcomes of flips, THTHH. It is crucial to recognize that

$$Pr(\text{even digit}) = 5/10 = Pr(H)$$

and

$$Pr(\text{odd digit}) = 5/10 = Pr(T).$$

Starting with the (randomly selected) 29th row of the Random Digit Table in the Appendix, we have the digits 09463 63823 which represent the outcomes HTHHT HTHHT. For our first experiment four flips, HTHHT, were needed to accumulate 3 heads. Starting our second experiment where the first one ended, we have the outcomes T HTHHT. So in the second experiment five flips, T HTHHT, were necessary.

Digits	09463	63823	29643	62401	06537	63918	52056	80333
H or T	HTHHT	HTHHT	HTHHT	HHHHT	HHTTT	HTTTH	THHTH	HTT
Experiment Number	1	2	3	4	5	6	7	
Number of Flips to Get 3 Heads	4	5	5	4	4	10	4	

Each vertical line notes the end of one experiment--after accumulating 3 H's a line was drawn. We start the next experiment with the next digit in the table and continue, digit by digit, until 3 more H's are obtained.

Seven experiments are not enough to give much confidence. The results of 100 experiments are given in the frequency histogram in Figure 2. The histogram indicates

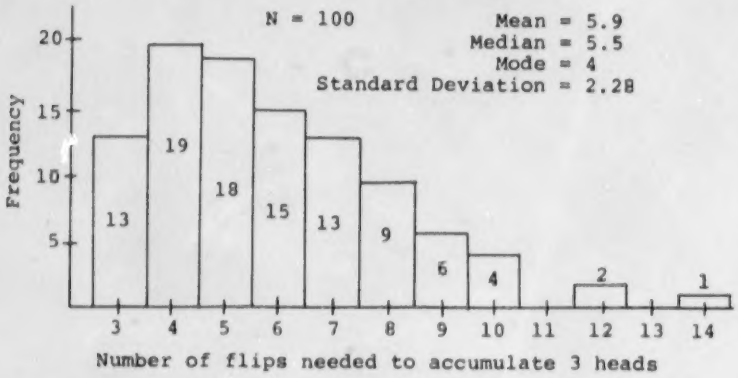

Figure 2. This histogram shows the results of 100 coin-flipping experiments. The goal in each experiment was to accumulate 3 heads. In 13 experiments, the first 3 flips gave 3 heads. In 19 experiments, it took 4 flips to get 3 heads. In 18 experiments, it took 4 flips to total 3 heads, and so on.

that most of the time (65%) it will take 6 or fewer flips to accumulate 3 heads. If we needed more accurate results, we could continue the simulation as long as necessary.

Exercises

2. Verify the values of the mean, median, mode, and standard deviation for the histogram in Figure 2.

3. Use the table of random digits in the Appendix to perform the experiment described above at least 20 times, and construct a frequency histogram of the experimental outcomes.

4. Use a simulation to approximate Pr(exactly 3 heads occur when 5 coins are flipped).

4. CORRESPONDENCE BETWEEN DIGITS AND EVENTS

If a cheating gambler had "fixed" the coin in the previous example so that Pr(H) = 0.6, then a different correspondence between the possible outcomes, H or T, and the possible digits, 0 to 9, would be necessary. One possible correspondence would be H $\longleftrightarrow$ {0,1,2,3,4,5} and T $\longleftrightarrow$ {6,7,8,9}. Then

$$Pr(H) = Pr(\{0,1,2,3,4,5\}) = 0.6$$

and

$$Pr(T) = Pr(\{6,7,8,9\}) = 0.4,$$

as required.

If Pr(H) = 0.63, the correspondence becomes only slightly more complex. Instead of using single digits, we can consider pairs of digits. The range of the pairs is 00 to 99, and one possible correspondence would be H $\longleftrightarrow$ {00 to 62} and T $\longleftrightarrow$ {63 to 99}. Then

$$Pr(H) = Pr(\{00 \text{ to } 62\}) = 0.63,$$

as required.

To simulate the rolling of a balanced die (6 sides), we could set up the correspondence

digit "1" in the table	$\longleftrightarrow$	side 1 on the die
digit "2" in the table	$\longleftrightarrow$	side 2 on the die
digit "3" in the table	$\longleftrightarrow$	side 3 on the die
digit "4" in the table	$\longleftrightarrow$	side 4 on the die
digit "5" in the table	$\longleftrightarrow$	side 5 on the die
digit "6" in the table	$\longleftrightarrow$	side 6 on the die
digits "7,8,9,0" in the table	$\longleftrightarrow$	No Event (the die rolled under the desk)

If the digits 7, 8, 9 and 0 are eliminated from the table the remaining digits are still randomly ordered, and each remaining digit occurs with relative frequency approximately 1/6.

Exercises

5. Use the correspondence just given, and the table of random digits in the Appendix, to perform at least 20 experiments to determine how many rolls of a die are usually necessary for the sum of the roll outcomes to exceed 6.

6. Set up a correspondence for the possible event outcomes and the random digits for
   (a) the rhythm method problem,
   (b) the lottery problem,
   (c) the drunkard problem, and
   (d) the grocery store problem.

5. RHYTHM METHOD -- WORKED

For the rhythm method example (page 1) an experiment could consist of keeping track of many women who use the method for a period of time. This is how the original "effectiveness" figures were compiled. But the Monte Carlo technique is much quicker and cheaper.

We will simulate the results for one woman and then repeat the experiment many times. An event will consist of a P (pregnancy) or an N (nonpregnancy) for a given year. An experiment will consist of a sequence of years until the occurrence of the first P. But first we need a correspondence between P and N and the digits. Since Pr(P) = 0.3 and Pr(N) = 0.7, one possible correspondence is P $\longleftrightarrow$ {0,1,2} and N $\longleftrightarrow$ {3 to 9}.

If we start with the (randomly selected) 16th digit of the 12th row of the Random Digit Table, we find

Digit	5	2	7	1	2	2	1	5	5	8	3	6	7	3	4	2	4	1	3	1	9	5	8	0	7	8	0	9	2	2	8	5	0	1	0
Event	N	P	N	P	P	P	P	N	N	N	N	N	N	N	N	P	N	P	N	P	N	N	N	P	N	N	P	N	P	P	P	N	N	P	P
Years to First P	2		2	1		1	1					9					2		2		4				3				2	1		3		1	1	

As in the coin example on page 5, each vertical line marks the end of one experiment. Fifteen is not a large number of experiments. The histogram shown in Figure 3 results from performing the experiment 100 times.

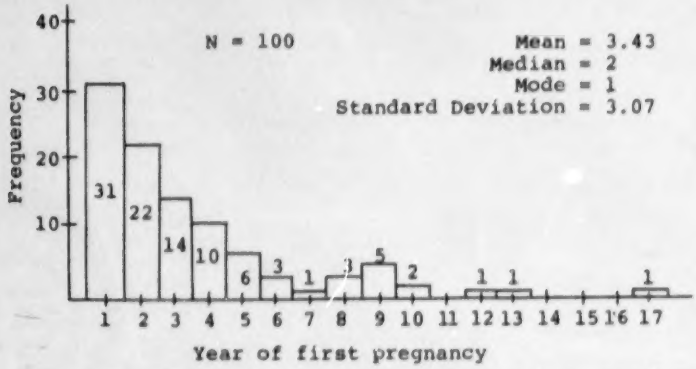

Figure 3. The results of 100 Monte Carlo simulations of the rhythm method experiment.

6. OUTLINE OF THE MONTE CARLO TECHNIQUE

I. List all possible outcomes for each event, e.g., H/T; P/N; or W/I/N.

II. Determine the probability of each outcome, e.g., 0.5/0.5; 0.3/0.7; or 0.5/0.4/0.1.

III. Determine subsets of the integers which have the same relative frequencies as the probabilities listed in II, e.g., {even}/{odd}; {0,1,2}/{3-9}; or {0,1,2,3,4}/{5,6,7,8}/{9}.

IV. Set up a correspondence between the outcomes and the subsets of integers, e.g., H $\longleftrightarrow$ {even}/T $\longleftrightarrow$ {odd}; P $\longleftrightarrow$ {0,1,2}/N $\longleftrightarrow$ {3-9}; W $\longleftrightarrow$ {0,1,2,3,4}/I $\longleftrightarrow$ {5,6,7,8}/N $\longleftrightarrow$ {9}.

V. Randomly select a starting point in the Table of Random Digits.

VI. Using each random number to represent the corresponding event outcome, perform the experiment and note the outcome.

VII. Repeat step VI until the desired confidence in the accuracy of the result is obtained.

Exercises

7. Using a certain tire manufacturing process Pr(defective tire) = 0.2, if you randomly select 5 tires as they come off the production line, use the Monte Carlo technique to estimate Pr(exactly 2 of the 5 are defective).

8. The Soggy Cereal Company includes a small toy in each box of Lumpy Lead cereal. There are 3 types of toys, and they are evenly distributed, one to a box. Follow the steps in the out-

line to determine the number of boxes of Lumpy Lead you should expect to have to buy in order to accumulate all 3 types of toys.

7. COMMENTS

7.1 The Name Monte Carlo

The use of random numbers is clearly vital to the Monte Carlo technique. To obtain these numbers, one could use a table like the one included in this module, or a computer (one was used to generate the table), or a suitably random physical device. A die could be used to generate a table of the digits 1 to 6; a roulette wheel could generate the numbers 1 to 36. The whole tecnnique has a strong flavor of gambling, as does much of probability theory.

During World War II, physicists working on the Manhattan Project encountered the problem of describing the behavior of neutrons in various materials. This problem had immediate applications to the construction of shielding and dampers for nuclear bombs and reactors. Direct experimentation would have been time-consuming and extremely dangerous. The basic data about the behavior of single neutrons were known, but there was no practical direct formula for calculating how a whole system would behave.

> "At this crisis the mathematicians John von Neumann and Stanislas Ulam cut the Gordian knot with a remarkably simple stroke. They suggested a solution which in effect amounts to submitting the problem to a roulette wheel. Step by step the probabilities of the separate events are merged into a composite picture which gives an approximate but workable answer to the problem."
>
> (Daniel McCraken, Scientific American, May 1955, p. 90)

The basic ideas of the technique had been around for a long time, but for its use in the secret work at Los Alamos, John von Neumann descriptively code named the method "Monte Carlo," after Europe's most famous gambling center.

7.2 The Number of Experiments

A discussion of the number of experiments necessary to attain a predetermined level of confidence in the final estimated answer would require too much space

and time as well as background in statistics. However, two general comments can be made:

(i) As the number of experiments increases, our confidence in the accuracy of the estimate should also increase. As a general rule, to double the accuracy of the result (cut the expected error in half), four times as many experiments are necessary.

(ii) If the outcome data are tightly bunched after many experiments (i.e., the standard deviation is small), we should have more confidence in the accuracy of our estimate than if the outcome data are scattered.

Confidence intervals for the mean and median of the outcome data can be found in most introductory statistics books. Introduction to Statistics by G. Noether and Statistics, a First Course by J. Freund both contain rules for determining the number of samples.

7.3 Use of a Computer

The repetition required by the Monte Carlo technique can be tedious and boring. But computers are very fast and not easily bored. Because of their speed, computers can often perform the "busy work" on very complex problems in reasonable amounts of time. Large numbers of experiments can be rapidly performed to detect small changes in probabilities, or to examine the effects of delicate changes in the system. You could compare different betting systems in Black Jack or roulette by having the computer play thousands of games using each strategy and comparing the results.

To use a computer for the "busy work," one must first have a good understanding of the ideas behind the technique, the "thinking" part. Before beginning to program a Monte Carlo experiment, it is a good idea to run a few experiments by hand to be certain of the procedure involved.

The vast majority of computers do not require the use of a random digit table since they are capable of generating their own random numbers.

8. ADDITIONAL EXERCISES

9. A grasshopper sits in the middle of a 7 foot long log. Each minute this grasshopper hops 1 foot to the right or left (with equal probability). How long do you expect it to remain on the log? (This is a 1-dimensional random walk.)

10. Based on at least 10 trials, preferably many more, estimate the solution to the Drunkard problem (page 1). You may find it easier to use graph paper to keep track of the intermediate steps and final stopping position.

Comment: This is sometimes called a random walk. It can also be used to study the mixing of gases or liquids by diffusion. Each labeled molecule in the figure can be treated as a "drunkard" and allowed to wander. Different step sizes would correspond to different temperatures.

B	B	B	B	A	A	A	A
B	B	B	B	A	A	A	A
B	B	B	B	A	A	A	A

Each molecule of Gases A and B is a "drunkard."

11. Based on at least 10 trials, preferably many more, estimate the solution to the Lottery problem (page 1).

Comment: This model can also be used to represent the selfreplication of a DNA strand in a medium which contains the four necessary components, represented by the letters A, T, C, G, in various proportions. One could study how a change in the proportions present will effect the time needed for replication.

9. PROBLEMS

The problems below are more complicated than the ones in the previous exercises. Read them and think about how they could be simulated, but don't perform the simulation unless you have access to a computer. Answers are given.

Problem 1

Determine the best batting order for a Mathball team. Mathball is a simplifed form of baseball and is played by the following rules:

Field:	3 bases; home plate, first base, and last base.
Team:	Each team has 5 players.
Game:	A game consists of 5 innings.
Inning:	A team bats in an inning until 3 outs are made.
Hitting:	A batter who gets a hit goes only to first base. (No doubles or home runs.)
Running:	A runner advances 1 base on a hit.

Your Team:	Batter	Average (probability of a hit)
	Tina	.200
	Ingrid	.500
	George	.200
	Elmer	.300
	Roger	.300

Problem 2A

Calculate that part of the area of the circle $x^2 + y^2 \leq 1$ which lies in the first quadrant ($x > 0$, $y > 0$). (This circle is centered at the origin and has radius 1.)

Comment: If we generate a large number of random points in the circumscribed square (see the diagram), then the ratio of the number of points in the quarter circle to the total number of points in the square will approximate the ratio of the area of the quarter circle to the area of the square. The area of the square is 1, so the area of the quarter circle is readily approximated.

$$\left(\frac{\text{points in quarter circle}}{\text{total points in the square}}\right) \approx \left(\frac{\text{area of quarter circle}}{\text{area of the square}}\right)$$

This technique of integration (finding area) is less efficient for people and computers in the two-dimensional case than several other approximate integration techniques (e.g. Riemann Sums or Simpson's Rule). But for multiple integration in higher dimensions variations of this Monte Carlo technique are competitive with other techniques and are frequently better.

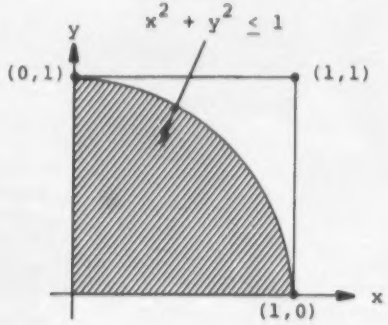

Figure 4. The area of a quarter circle may be estimated by the Monte Carlo technique described in Problem 2A.

Problem 28

Calculate that part of the volume of the sphere $x^2 + y^2 + z^2 \leq 1$ which lies in the first octant ($x > 0$, $y > 0$, $z > 0$). See the diagram below.

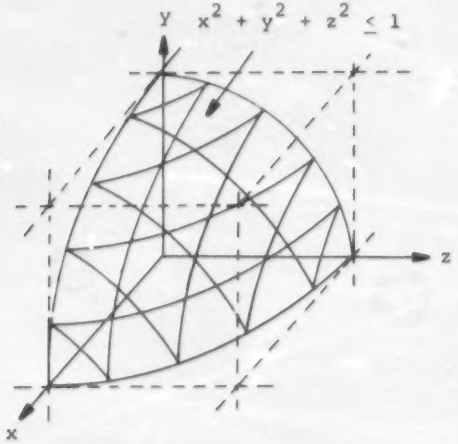

Figure 5. Monte Carlo techniques may be used effectively to calculate volumes.

10. MODEL EXAM

Directions: For each problem:

(1) Set up the correspondence between the possible experiment outcomes and the digits,
(2) perform the experiment at least 20 times using the table of random digits, and
(3) draw a frequency histogram of the experimental results and compute the mean.

1. If you randomly select two 1-digit numbers (0,1,2,...,9), what is the expected distance between them?

2. If you decide to have a family of 4 children, what is the probability that the resulting family is 2 boys and 2 girls? (Assume Pr(boy) = Pr(girl) = 0.5.)

3. The Soggy Cereal Company is running out of whistles to put into their cereal boxes, so they are putting shoulder patches in 70% of the boxes and whistles in the remaining 30%. Assuming that the boxes are well shuffled, about how many boxes should you expect to have to buy in order to get both prizes?

4. If you sit down for lunch with 5 other people, what is the probability that at least 2 of the 6 people at the table were born under the same astrological sign? (Assume that a person is equally likely to have been born under each of the 12 signs.)

5. In a popular board game, the attacking player roles 2 dice and the defending player roles only 1. The attacker wins if the higher of his 2 dice is <u>larger</u> than the number shown on the defender's die. Calculate the probability that the attacker wins.

6. Use the Monte Carlo technique to estimate the area between the curve $y = x^2$ and the x-axis for $0 \leq x \leq 1$. (See the diagram on the front cover.)

11. ANSWERS TO SOME EXERCISES AND PROBLEMS

Exercise 3

Your frequency histogram should have the same general shape as the histogram on page 4, although yours will probably be more jagged.

Exercise 4

The exact probabilities (using the Binomial Formula) are

P(exactly 0 heads in 5 flips) = 0.031
P(exactly 1 head in 5 flips) = 0.156
P(exactly 2 heads in 5 flips) = 0.312
P(exactly 3 heads in 5 flips) = 0.312 ***
P(exactly 4 heads in 5 flips) = 0.156
P(exactly 5 heads in 5 flips) = 0.031

Your estimate should be close to 0.312.

Exercise 5

These are the results of 2000 experiments.

Number of Rolls Needed to Exceed 6	Frequency	Percent
1	0	0.00%
2	1134	56.70%
3	673	33.65%
4	173	8.65%
5	19	0.95%
6	1	0.05%

Exercise 6

(a) See Section 5.

(b) "W" ↔ {0,1,2,3,4}; "I" ↔ {5,6,7,8}; "N" ↔ {9}

(c) North ↔ {00 to 24}; South ↔ {25 to 49};
East ↔ {50 to 74}; West ↔ {75 to 99}.

(d) 0 new shoppers ↔ {0,1,2}
1 new shopper ↔ {3,4,5,6}
2 new shoppers ↔ {7,8,9}.
There are other correct correspondences for each of these.

Exercise 7

P(exactly 2 are defective) = 0.2048. Your estimate should be close to this value.

Exercise 8

Based on the results of 1000 experiments (shoppers), the mean number of boxes needed to acquire all 3 different toys was 5.6.

P(3 boxes were required) = 0.222
P(4 boxes were required) = 0.222
P(5 boxes were required) = 0.173
P(6 boxes were required) = 0.123
.
.
.

One of the 1000 shoppers needed 28 boxes to get all 3 toys.

Exercise 9

These are the results for 1000 experiments. Mean Number of Hops = 15.8, Median Number of Hops = 12. In this situation the maximum number of hops observed before the grasshopper fell off the log was 68, but the bug could occasionally stay on much longer.

Exercise 10

These are the results for 2000 experiments.

	Mean	Standard Deviation
5 Random Steps	2.03	0.96
10 Random Steps	2.78	1.43

Doubling the number of steps does not double the expected distance from the lamp--some directions tend to bring the drunkard back to the lamp. After one step the drunkard will be exactly 1 unit from the lamp, but after two steps the expected distance will be $(0 + 2 + \sqrt{2} + \sqrt{2})/4 = 1.207$ units.

Exercise 11

These are the results for 1000 ticket buyers.

Word	Mean Number of Tickets Needed	Median	Standard Deviation
"WIN"	11.07	8	9.40
"IWIN"	11.37	8	8.80

In this simulation one ticket buyer needed 78 tickets.

Problem 1

Runs per game averages are given for several batting orders. Each average is based on 500 games using the order.

Batting Order (by bat. avg.)	Mean Number of Runs Per Game	Standard Deviation
0.5/0.3/0.3/0.2/0.2	4.732	2.96
0.3/0.5/0.3/0.2/0.2	4.496	2.86
0.3/0.3/0.5/0.2/0.2	4.586	2.89
0.2/0.2/0.3/0.3/0.5	4.166	2.92
0.3/0.3/0.2/0.5/0.2	4.434	3.02

A more sophisticated model might include doubles and home runs as well as base stealing and double plays.

Problem 7

Exact area of 1/4 circle = $\pi/4$ = 0.785398.
Exact volume of 1/8 sphere = $\pi/6$ = 0.523599.

Number of Points	Estimated Area (error)		Estimated Volume (error)	
10	0.9	(.115)	0.5	(0.024)
20	0.6	(.185)	0.35	(0.174)
100	0.74	(.045)	0.53	(0.006)
1000	0.776	(.0094)	0.527	(0.0034)
10000	0.7827	(.0027)	0.5239	(0.0003)

12. ANSWERS TO MODEL EXAM

1. The mean distance between 2 randomly selected digits is 3.30. Your estimate should be close to that.

2. Exact probabilities (using the Binomial Formula) are

   Pr(4 boys, 0 girls) = Pr(0 boys, 4 girls) = 0.0625
   Pr(3 boys, 1 girl) = Pr(1 boy, 3 girls) = 0.25
   Pr(2 boys, 2 girls) = 0.375

   Your estimated value should be close to 0.375.

3. Based on 4000 experiments (shoppers), the mean number of boxes needed was 3.78, the median was 3.

   Pr(need only 2 boxes) = 0.42
   Pr(need only 3 boxes) = 0.21
   Pr(need only 4 boxes) = 0.1218

4. Below are the exact probabilities for various numbers of persons at the table.

Number at Table	P(at least 2 share a sign)	Number at Table	P(at least 2 share a sign)
2	0.007	8	0.954
3	0.236	9	0.985
4	0.427	10	0.996
5	0.618	11	0.9996
**6	0.777**	12	0.99996
7	0.889	13	1.0

Your estimate should be close to 0.777.

5. Based on 1000 attacks, the attacker won 582 times. Your estimate should be close to 0.582.

6. The exact area, using elementary calculus, is 1/3. (See Problem 2A.) Estimates will vary.

APPENDIX:
RANDOM NUMBERS

A1. Properties and Tests

The "Random Digits" generated by computers are not truly random, but are usually determined by some procedure from a previous number. However, a "good" procedure will generate numbers with properties which truly random numbers would have. The most basic and desirable of these properties are:

A1.1 Uniform Distribution: Each digit occurs with about the same frequency. Sometimes pairs and triples of digits are also used to test for uniform distribution.

A1.2 Independence: The digits do not appear to follow any regular pattern. Since there are so many possible patterns which could occur, it is impossible to test for all patterns. However, it is possible to test for some of the more obvious ones.

Up/Down Test: How often is the next number larger, smaller, or the same as the previous number in the table? For truly random numbers, about 10% of the time the next digit should be the same as the previous digit, about 45% of the time (half of the remaining 90%) it should be smaller, and about 45% of the time it should be larger.

Cycles: How long before the digits start to repeat in the same order? How long until the digits cycle? This is usually very difficult to determine just by examining the procedure or the resulting table. A "good" procedure takes a very long time before starting the cycle.

A technique for generating "random" numbers can be found in "Methods of Random Number Generation" by Edwin Landauer in The Two-Year College Mathematics Journal, November 1977, pages 296-303.

A.2 Use of the Random Digit Table

Starting Point: If you want to get "fancy" you could use a spinner or a pair of dice to generate two random numbers--the first number to be the starting row, and the second to be the starting digit in that row (e.g., the pair 3,6 would direct you to start with the 6th digit of the 3rd row).

But most people use the "blind stab" technique; close your eyes, point to a point on the page and start there.

Continuing: Since the digits in the table are already in random order it is not necessary to select a new starting point for each experiment--simply start the next experiment with the next digit in the table.

A.3 10,000 Random Digits

Frequency of Digit	Page 1	Page 2	Page 3	Page 4	Total	Total Percent
0	249	252	253	243	997	9.97
1	237	266	257	269	1029	10.29
2	281	241	242	247	1011	10.11
3	270	227	252	244	993	9.93
4	253	256	250	226	985	9.85
5	257	244	249	256	1006	10.06
6	229	260	245	250	984	9.84
7	229	268	255	250	1002	10.02
8	256	243	225	285	1009	10.09
9	239	243	272	230	984	9.84
Same	251	239	255	248	993	9.93
up	1125	1133	1141	1131	4530	45.30
Down	1124	1128	1104	1121	4477	44.77

82698 26610 90511 08055 80364 70233 91451 34528 30357 27456
93680 27051 67692 57437 08779 81065 50586 20621 28296 43353
45153 17985 74725 08526 09220 89778 59814 02387 78112 16035
65055 40547 20834 50243 23998 59708 12313 89349 25103 43682
80863 76681 73173 48970 91202 81344 89446 60285 12653 95567

65704 35329 80233 67505 22518 58994 63968 79316 53447 65610
16862 82356 69963 61171 96043 56593 73637 82198 51634 71363
76048 34462 57543 98743 80838 42517 42094 98970 07496 22223
92003 32221 39595 99113 43596 90842 87684 80098 54888 32782
74244 90661 80795 20305 92055 54532 99534 34660 41569 88305

38128 35924 55245 97971 52694 92422 15875 18971 20058 78333
33729 56998 99535 52712 21558 36734 24131 95807 80922 85010
63971 68875 13322 07349 73991 41072 31419 29611 10297 85465
57653 56330 22804 71402 62635 33217 85828 69039 77095 57063
36395 30423 96224 53481 23420 44921 30883 56083 32038 63699

90543 52660 09346 76795 89783 87944 92379 34576 18055 67418
58133 19098 70130 16092 43843 80508 96387 42270 35335 18264
57487 88972 50914 65331 87902 42601 85407 19867 77391 48159
77128 23219 48346 02047 63984 66444 83317 40167 39020 00798
13964 87042 24341 25448 30779 30472 92064 71532 47311 33061

03114 30226 65252 72519 11706 72966 95952 93649 64857 57621
41182 02953 20581 46556 03312 24241 54804 29809 04113 75128
94953 59747 35056 70403 17822 04416 08601 45680 69568 35183
26528 96679 08165 34005 90199 48983 99761 51229 31275 27314
71479 66012 23245 49574 10116 41521 06750 29164 63007 55902

42292 82996 86159 79513 84410 45582 38596 55311 04895 13515
63234 72661 32908 22815 30490 01502 52419 97075 95007 03410
95770 34807 06273 59221 42470 68812 28923 28313 16271 06813
09463 63823 29643 62401 06537 63918 52056 83389 86422 88943
37271 74277 85283 81867 66660 40978 [illegible]906 50846 32802 18984

29627 48227 75458 13027 68341 24267 89088 40988 53103 79923
64074 46280 63010 53561 12276 25624 43287 38239 56965 03913
41630 85293 87811 97757 34504 72791 18594 21759 58785 72898
49567 12521 70419 45853 84408 65065 47690 61921 43879 15782
16466 91260 29140 78385 54921 16937 33072 60675 18101 77288

78056 96364 45121 63361 65742 03964 01998 36442 97701 85267
53747 76678 84504 88985 58473 24216 20720 99730 44223 81412
35832 52303 07091 97235 22488 73307 63024 42020 82151 67453
00316 98955 28410 62443 65885 14166 61937 71899 05764 73820
03281 21503 85071 43185 37724 27177 54710 86306 83226 05223

88027 73621 54167 61642 15908 74027 32009 40957 00489 50941
58412 57958 10784 91459 05057 09259 04821 48798 94313 22552
26713 91036 37943 39567 35577 92419 27216 48996 88339 11204
74903 84443 30195 38568 91675 53618 40088 24647 70893 99334
63137 40699 42046 10281 57445 67771 00976 20883 79039 54049

45053 55031 75934 73950 89878 20357 03257 14471 24981 42633
39121 28849 82954 36481 10444 85221 55466 27512 03441 57984
45968 12528 55047 03065 63942 45232 22368 05620 22057 13135
62420 23116 23984 50249 42438 47611 68085 84966 08318 18250
69928 18998 17186 08202 28286 60156 27066 95713 47429 64033

68420 91594 18774 99086 97471 11096 83934 54694 99278 53366
10766 65860 45180 01167 45771 87610 05272 85867 82672 68059
59782 87239 59260 65113 59876 10642 79247 45118 65702 96858
50136 54869 77626 25256 27837 49592 37705 01488 05843 88203
24186 44144 99986 26937 10126 47675 36747 22790 65792 22751

32149 32697 40420 71863 11583 02864 25198 15551 90871 10326
71152 46507 83616 93181 68659 77281 18518 27371 68757 18253
98480 39095 31712 53194 51924 06287 57890 12455 01641 85052
88139 98726 64244 13517 03796 92669 46544 77797 63147 47743
83133 03717 59230 24429 89823 69684 22210 56398 71136 22323

53546 43190 65854 17069 11174 88317 85699 13809 67271 94418
48162 65787 83194 80075 59176 32700 38999 41747 54312 69993
84814 97006 58212 24471 00035 20523 67758 63351 69789 62877
17462 44657 39043 05410 13946 13306 73265 42812 81182 47604
69490 13754 57511 73595 60986 91695 36815 68175 46810 17198

44817 89371 87318 64743 96118 62417 34657 53990 18410 85309
78895 96529 26425 94164 79378 85802 35855 47916 07173 33690
61302 09781 24426 50261 49587 09675 11506 48489 86292 32713
37458 11722 96040 26021 43539 68552 16742 38625 30907 03649
41800 34521 69828 47464 16216 72943 00330 93677 38492 08708

14708 41640 22349 89030 62190 14042 13371 62037 33843 04555
08497 13633 90824 32538 31091 81954 91588 80743 90094 31006
19085 81561 47008 01014 23479 26661 70725 77994 95119 28515
05856 76772 94061 87862 56015 86653 82671 06105 50992 52662
38818 06893 03319 32736 25017 70617 49879 73150 12355 30007

74183 52870 70880 87765 25043 05881 69958 33040 06060 99228
71626 80724 43948 82019 56251 40368 63507 10557 74890 25340
60240 60570 16600 16414 70969 69191 33937 47968 29374 93538
05759 07744 12089 90706 94402 10132 75795 27739 88054 67702
24124 49735 10951 60217 65867 16628 80069 31145 42728 72525

49727 52958 52316 95660 66210 64217 20436 32849 24576 40591
81607 30289 71071 02563 32613 66914 00753 60781 09185 76051
60087 18737 05805 27414 42912 77982 68504 67410 87694 49195
93765 51974 86490 26739 16100 58912 99557 29283 52530 14750
87941 61498 16658 05112 29020 34744 25975 59405 88830 48603

04392 42554 34738 97944 69423 22576 36792 02929 35868 45485
77057 75328 28431 68407 96972 18792 53721 48557 94522 22621
56462 32852 12144 00576 24336 97318 40797 73018 91053 70169
39735 44807 82227 46221 60117 04324 05759 45700 13299 65149
83050 41721 29138 38823 34923 93301 98190 27037 72070 56465

62314 32142 24102 69218 20065 76827 04831 07796 34291 50754
38950 28005 63258 67274 33980 68269 89313 38086 13712 95206
52433 45631 05969 75331 64046 29691 13143 55478 53127 64545
64146 82545 12934 80945 73589 33866 10603 70451 66164 44446
09412 55663 59584 83213 69608 28923 66469 17481 71246 89910

04503 72085 84585 52396 99506 26123 61681 38641 65181 34728
91891 38326 29940 45907 45271 45001 97684 24776 73079 91367
91115 19370 48461 77755 42845 87704 48785 96845 62677 84985
34907 68127 68011 77047 78265 87344 65971 04187 40044 73874
99856 31340 10282 20389 65561 25532 40236 48359 90606 33979

27037 72070 56465 62314 32142 24102 69218 20065 76827 04831
07796 34291 50754 38950 28005 63258 67274 33980 68269 89313
38086 13712 95206 52433 45631 05969 75331 64046 29691 13143
55478 53127 64545 64146 82545 12934 80945 73589 33866 10603
70451 66164 44446 09412 55663 59584 83213 69608 28923 66469

17481 71246 89910 04503 72085 84585 52396 99506 26123 61681
38641 65181 34728 91891 38326 29940 45907 45271 45001 96784
24776 73079 91367 91115 19370 48461 77755 42845 87704 48785
96845 62677 84985 34907 68127 68011 77047 78265 87344 65971
04187 40044 73874 99856 31340 10282 20389 65561 25532 40236

48359 90606 33979 09262 40436 09883 01575 68238 27119 17924
07467 09580 91949 40502 88651 71376 75607 04357 90371 55872
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42665 91326 49901 98208 32107 71929 80351 72674 21477 92890
73228 68631 21125 69520 54548 27554 24653 07923 80316 20491
42600 44409 28867 46699 87567 93070 24929 47224 50096 61129
47795 04434 64756 94677 32392 77386 10028 87136 67855 36325
00674 55308 78280 52420 56172 35985 32502 07917 60875 32698

96948 49982 34663 81020 08403 41764 80806 98992 33011 83328
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51521 72704 16153 91634 00414 39927 18382 84077 38849 69007
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07757 10548 60790 46231 67260 34411 98285 47594 71957 54096
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46250 93850 94388 07478 96634 32036 33138 70445 38034 54464
43158 43643 57121 76627 71454 97036 72365 13356 92051 73963

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11660 88785 25096 15556 68804 51431 08324 46281 59901 75352
19106 09665 36862 59300 57953 89717 41554 32228 99023 79115
20960 45950 54946 44967 23065 26049 10121 79641 96642 85528
65018 57366 60878 71150 17045 63518 79555 13778 44533 72487

96558 42015 54031 77461 44942 96934 36057 54767 77631 94379
77942 86764 27579 66873 76015 49750 57561 52388 31914 15033
81681 81749 30071 81901 82813 91797 71713 39843 72933 04286
52570 35966 06913 47050 51782 83549 35859 60841 87978 46089
84425 73503 28578 11685 16815 15978 71920 60815 53895 16900

25535 79807 68457 14232 78851 32810 76576 39616 18218 22350
48138 11077 43245 25517 98105 79268 41262 05899 99484 52920
11627 66583 65057 70008 51932 73879 68870 82455 48444 10163
31856 07704 91883 61734 18850 65907 98368 11990 52967 05709
37440 43130 86051 46031 06637 34541 62925 29648 94378 56422

35746 62282 17094 04250 27396 04887 56023 53245 92661 85005
97804 87773 82534 15870 49848 80953 31171 43955 80603 27364
91175 98163 28490 92192 06701 09367 32946 69758 00663 96301
38302 33327 44826 09816 96198 79200 63473 66838 34831 50093
09180 07928 29973 81575 15131 51129 60741 16445 36119 49166

55666 50999 13101 18800 83267 68427 99139 89986 40883 22136
29038 91236 46064 39305 25733 73713 58019 81522 70278 45161
84577 95714 88417 56123 14788 03531 11364 15741 54252 80109
13885 72669 93446 02807 17864 56620 29762 51934 65284 97767
31558 13888 11120 13444 25443 62756 99367 65893 82758 71926

51899 52677 07831 47908 40581 22500 91302 52878 90316 06545
20509 72146 63389 23334 13954 80998 83254 22812 38984 72714
03217 22731 94040 21712 76040 63934 88640 12379 25720 75254
88124 29703 13149 86345 03022 36374 74372 16250 40762 29498
41194 15611 78675 81355 85500 01372 06285 87351 32462 15277

20934 10776 24867 85067 30259 57928 13688 40697 12574 80184
44809 12889 67228 84743 49216 38306 81830 34922 71881 37172
62731 23281 74601 51960 90468 28746 80548 07593 71242 10987
54498 35814 84711 86001 27335 75586 48296 10307 33053 45522
20236 85252 25616 86696 28140 76975 56010 17620 31577 78843

50128 86377 67646 45256 80934 14582 93088 82511 68214 30086
18022 72895 58151 83733 21216 80519 92430 82367 29961 00058
25245 68730 00615 18502 99241 78775 52898 96479 47632 23135
64830 21037 12282 53755 34260 95694 58986 05376 69326 73447
18707 81752 39539 50986 24542 48826 40438 01398 30363 17431

82453 18982 80155 17210 62812 68135 25603 88137 58265 34246
00981 58914 69021 10403 47073 57405 83520 62114 19764 14452
08392 82666 30928 69790 21875 57349 73750 41290 32488 49351
51013 70767 27662 54188 27703 95973 97846 46031 44783 25761
37469 09296 35058 19965 47830 17051 53582 78327 05699 66906

32261 50793 29185 29158 35977 65910 96837 80985 95785 62837
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Emmy Noether, 1882-1935

Auguste Dick

translated by Heidi Blocher

A brief biography of Emmy Noether has been a "well-known secret" among German reading mathematicians for over ten years. It is Auguste Dick's *Emmy Noether, 1882-1935*, published as the thirteenth supplement to the journal, *Elemente der Mathematik*. The famous poster 'Men of Mathematics' presented at the World's Fair in New York in 1964 says about her:

"Emmy Noether, daughter of the mathematician Max, was often called "Der Noether", as if she were a man. Yet her Gottingen professorship carried no salary and Hilbert had to fight to get her—a woman—in at all. She was fat, rough, and loud, but so kind, humorous and sociable that all who knew her loved her. With the Nazi rise to power she left for the United States.

"Emmy's early work on invariants gave no hint that she would become one of the creators of abstract axiomatic algebra. She developed the axiomatic theory of ideals, introducing the ascending chain condition, gave a unified theory of non-commutative algebras and their representations, defined the 'verschränkte Produkt' and, with Brauer and Hasse, proved that every simple algebra over an algebraic number field is cyclic."

Auguste Dick's biography, complemented by obituaries and memorials written by Pavel Sergeevic Alexandroff, B.L. van der Waerden and Hermann Weyl captures the spirit and essence of her life and work. This translation has been amended with some previously unpublished photographs and comes as a timely tribute to one of the greatest mathematicians of our age.

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